

Oscillating heat kernels on ultrametric spaces

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Abstract

Let (X, d) be a proper ultrametric space. Given a measure m on X and a function $B \mapsto C(B)$ defined on the collection of all non-singleton balls B of X , we consider the associated hierarchical Laplacian $L = L_C$. The operator L acts in $\mathcal{L}^2(X, m)$, is essentially self-adjoint and has a pure point spectrum. It admits a continuous heat kernel $\mathfrak{p}(t, x, y)$ with respect to m . We consider the case when X has a transitive group of isometries under which the operator L is invariant and study the asymptotic behaviour of the function $t \mapsto \mathfrak{p}(t, x, x) = \mathfrak{p}(t)$. It is completely monotone, but does not vary regularly. When $X = \mathbb{Q}_p$, the ring of p -adic numbers, and $L = \mathcal{D}^\alpha$, the operator of fractional derivative of order α , we show that $\mathfrak{p}(t) = t^{-1/\alpha} \mathcal{A}(\log_p t)$, where $\mathcal{A}(\tau)$ is a continuous non-constant α -periodic function. We also study asymptotic behaviour of $\min \mathcal{A}$ and $\max \mathcal{A}$ as the space parameter p tends to ∞ . When $X = S_\infty$, the infinite symmetric group, and L is a hierarchical Laplacian with metric structure analogous to \mathcal{D}^α , we show that, contrary to the previous case, the completely monotone function $\mathfrak{p}(t)$ oscillates between two functions $\psi(t)$ and $\Psi(t)$ such that $\psi(t)/\Psi(t) \rightarrow 0$ as $t \rightarrow \infty$.

1 Introduction

At the centre of this paper stands the study of the on-diagonal asymptotics of the heat kernels for isotropic Markov processes on homogeneous ultrametric spaces. We focus on the precise computation of its periodic oscillations and the corresponding background.

Motivation for this paper comes from various sides. One is the interest in properties of random walks on infinite groups, in particular asymptotics of

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transition probabilities. While there is a great body of rigorous mathematical work on this subject regarding finitely generated groups, here we have drawn inspiration from the work on *infinitely generated* ones. This goes back to Darling and Erdős [13] and was further pursued by various authors, among which Flatto and Pitt [18], Cartwright [12], Lawler [28] and Brofferio and Woess [11]. Most notably, [12] is to our knowledge the first to have exhibited evidence of oscillatory behaviour for return probabilities in such a case. A similar phenomenon was found for random walks on the infinite graph versions of certain fractals: on the Sierpinski-graphs in arbitrary dimension, precise asymptotics and periodic oscillations were exhibited by Grabner and Woess [19], [36, §16], later generalised to a larger class of self-similar graphs in [24].

The above-mentioned random walks on infinitely generated groups can be described as isotropic Markov processes on ultrametric spaces in discrete as well as continuous time. In Theoretical Physics, such spaces are also known as *hierarchical lattices*, proposed by F. J. Dyson in his famous paper on the phase transition for the **1D** ferromagnetic model with long range interaction [15]. The corresponding *hierarchical Laplacian* L was studied in a variety of papers, see e.g. [10], [2], [26], [27], [23], [25], [29], [31], [14]. It is the generator of a Markov process, which in the situation of discrete time and space is a random walk on an infinitely generated group as mentioned above.

A systematic study of isotropic Markov semigroups defined on an ultrametric measure space (X, d, m) has been carried out by Bendikov, Grigor'yan and Pittet in [5] (where X is discrete) and by the same authors plus Woess in [6] (where X may contain both isolated and non-isolated points). The approach of those papers puts the previous work into a well accessible framework which has lead to a good understanding and various new results. One of the key features is that for the Markov generator (that is, the hierarchical Laplacian), one obtains a full description of the spectrum, which is pure point, along with a complete system of compactly supported eigenfunctions.

On the basis of this spectral decomposition, we study here the asymptotics of the diagonal elements of the heat kernel (transition kernel). We now display the setup and basic features in more detail.

General setup and basic facts

The state space of our processes is assumed to be a proper ultrametric space (X, d) . Recall that proper means that closed balls are compact. Our metric d must satisfy the ultrametric inequality

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \quad (1.1)$$

A basic consequence is that each open ball is compact. Furthermore, for each $x \in X$, the set of distances $\{d(x, y) : y \in X\}$ is countable and does not accumulate in $(0, \infty)$. Two balls are either disjoint or one is contained in the other. The collection of all balls with a fixed positive radius forms a countable partition of X , and decreasing the radius leads to a refined partition: this is precisely the structure of a “hierarchical lattice” as in the old papers, going back to [15].

The analysis undertaken here is of interest in the case when X is non-compact, which will be assumed throughout this paper.

Our setup is based on the following two ingredients. The first is a Radon measure m on X such that $m(X) = \infty$ and $m(B) > 0$ for each closed ball which is not a singleton, and $m(\{a\}) > 0$ if and only if a is an isolated point of X . Now let \mathcal{B} be the collection of all balls with $m(B) > 0$. Then each $B \in \mathcal{B}$ has unique *predecessor* or *parent* $B' \in \mathcal{B} \setminus \{B\}$ which contains B and is such that $B \subseteq D \subseteq B'$ for $D \in \mathcal{B}$ implies $D \in \{B, B'\}$. In this case, B is called a *successor* of B' . By properness of X , each non-singleton ball has only finitely many (and at least 2) successors. Their number is the *degree* of the ball.

The second ingredient is a *choice function* $C : \mathcal{B} \rightarrow (0, \infty)$ which must satisfy, for all $B \in \mathcal{B}$ and all non-isolated $a \in X$,

$$\begin{aligned} \lambda(B) &= \sum_{D \in \mathcal{B} : D \supseteq B} C(D) < \infty, \quad \text{and} \\ \lambda(\{a\}) &= \sum_{B \in \mathcal{B} : B \ni a} C(B) = \infty. \end{aligned} \tag{1.2}$$

Let \mathcal{F} be the set of all locally constant functions having compact support. Given the space X , the measure m and the choice function C , we define (pointwise) the hierarchical Laplacian L_C : for each f in \mathcal{F} and $x \in X$,

$$L_C f(x) := \sum_{B \in \mathcal{B} : B \ni x} C(B) \left(f(x) - \frac{1}{m(B)} \int_B f \, dm \right). \tag{1.3}$$

The present work builds upon the following facts concerning L_C and its spectrum, which were proved in [5] (discrete space) and in full generality in [6]. The operator (L_C, \mathcal{F}) acts in $\mathcal{L}^2 = \mathcal{L}^2(X, m)$. It is symmetric and admits an \mathcal{L}^2 -complete system of eigenfunctions $\{f_B : B \in \mathcal{B}\}$, given by

$$f_B = \frac{\mathbf{1}_B}{m(B)} - \frac{\mathbf{1}_{B'}}{m(B')}. \tag{1.4}$$

The eigenvalue corresponding to f_B depends only on B' and is $\lambda(B')$, as given in (1.2). Since all f_B belong to \mathcal{F} and the system $\{f_B : B \in \mathcal{B}\}$ is complete

in \mathcal{L}^2 we conclude that (L_C, \mathcal{F}) is an essentially self-adjoint operator in \mathcal{L}^2 . By a slight abuse of notation, we shall write (L_C, Dom_{L_C}) for its unique self-adjoint extension. For all of this we refer to [7], [6]. See also the related papers [22], [23], [30].

Observe that to define the functions $C(B)$, $\lambda(B)$ and in particular the operator (L_C, Dom_{L_C}) we do not need to specify the ultrametric d . What is needed is the family of all balls which evidently can be the same for two different ultrametrics d and d' , via a feasible change of the diameter function. On the other hand, given the data (X, d, m) and choosing the function

$$C(B) = \frac{1}{\text{diam}(B)} - \frac{1}{\text{diam}(B')}, \quad (1.5)$$

where $B \in \mathcal{B}$ and B' is the predecessor of B , we obtain the hierarchical Laplacian (L_C, Dom_{L_C}) having eigenvalues of the form

$$\lambda(B) = \frac{1}{\text{diam}(B)}, \quad B \in \mathcal{B}. \quad (1.6)$$

We will refer to the resulting operator (L_C, Dom_{L_C}) as the *standard* hierarchical Laplacian associated with (X, d, m) . Vice versa, even if we start with an *a priori* ultrametric d on X , any choice function satisfying (1.2) induces the possibly different *intrinsic* metric d_* via (1.5) and (1.6): one sets

$$\text{diam}_*(B) = \frac{1}{\lambda(B)}, \quad B \in \mathcal{B}, \quad (1.7)$$

so that for distinct $x, y \in X$, one has $d_*(x, y) = \text{diam}_*(B)$, where B is the smallest ball containing both elements. By construction, the collection of d_* -balls coincides with the collection of d -balls, so that both metrics induce the same family of hierarchical Laplacians when varying the choice function C .

The general theory developed in [5] and [6] applies here. In particular, under mild assumptions – which hold in the situations that we consider here – the semigroup (P^t) induced by L_C admits a continuous heat kernel $\mathfrak{p}(t, x, y)$ with respect to m .

It has an integral representation in terms of the *spectral function*

$$N(x, \tau) = 1/m(B_*(x, 1/\tau)), \quad (1.8)$$

where $B_*(x, r) = \{y \in X : d_*(x, y) \leq r\}$, see [6, Def. 2.8.]. In particular,

$$\mathfrak{p}(t, x, x) = t \int_0^\infty N(x, \tau) \exp(-t\tau) d\tau. \quad (1.9)$$

Since $\tau \mapsto N(x, \tau)$ is a step function, the integral reduces to an infinite sum. For the analysis undertaken in this paper, we require the following to hold.

Definition 1.1 The ultrametric measure space (X, d, m) and the hierarchical Laplacian L_C are called *homogeneous*, if there is a group of isometries of (X, d) which

- acts transitively on X , and
- leaves both the reference measure m and the function $C(B)$ invariant.

The first assumption implies that (X, d) is either discrete or perfect. Basic examples which we have in mind are

1. $X = \mathbb{Q}_p$ – the ring of p -adic numbers, where $p \geq 2$ (integer).
2. $X = \bigoplus_{j=1}^{\infty} \mathbb{Z}(p)_j$ – the direct sum of countably many copies $\mathbb{Z}(p)_j$ of the additive group $\mathbb{Z}(p) = \mathbb{Z}/(p\mathbb{Z})$.
3. $X = S_{\infty}$ – the infinite symmetric group, that is, the group of all permutations of the positive integers that fix all but finitely many elements.

In this setting, our main goal is to study the asymptotic behaviour of the function $t \mapsto \mathbf{p}(t) = \mathbf{p}(t, x, x)$ as t tends to 0 or to ∞ ; it does not depend on x by homogeneity. Our study was inspired by the results of [12], [19], [28] and [11]. The papers [4] and [8] are direct forerunners of the present work.

Let us describe the main body of the paper. In Section 2 we discuss in more detail the consequences of the homogeneity assumptions of Definition 1.1. In particular, we show that the on-diagonal heat kernel $\mathbf{p}(t)$ cannot vary regularly. In Section 3 we consider $X = \mathbb{Q}_p$, the ring of p -adic numbers. We study the operator \mathfrak{D}^{α} , the p -adic fractional derivative of order $\alpha > 0$. This operator was first considered by Taibleson [32] as a spectral multiplier on \mathbb{Q}_p as well as on \mathbb{Q}_p^d . In relation to the concept of p -adic Quantum Mechanics, it was introduced under the above name by V.S. Vladimirov [33], [34], [35]. The operator \mathfrak{D}^{α} is the most typical example of a homogeneous hierarchical Laplacian. We show that the associated on-diagonal heat kernel on \mathbb{Q}_p has the form $\mathbf{p}_{\alpha}(t) = t^{-1/\alpha} \mathcal{A}(\log_p t)$, where $\mathcal{A}(\tau)$ is a non-constant strictly positive continuous α -periodic function, and that, as p tends to infinity,

$$\max \mathcal{A}(\tau) \rightarrow (e\alpha)^{-1/\alpha} \quad \text{and} \quad \min \mathcal{A}(\tau) \sim \frac{(\ln p)^{1/\alpha}}{p}.$$

In Section 4 we briefly explain the equivalent approach in terms of isotropic Markov processes. In particular, we focus again on the homogeneous situation, where we get isotropic random walks with discrete as well as continuous

time on ultrametric groups. We consider there case where the group is the direct sum of a countable family of copies of one finite group.

Finally, in Section 5 we consider a class of homogeneous hierarchical Laplacians acting on $\mathcal{L}^2(X, m)$, where $X = S_\infty$ is the infinite symmetric group and m is the counting measure. We show that the function $\mathbf{p}(t)$ oscillates between two functions $\psi(t)$ and $\Psi(t)$, where $\Psi(t)/\psi(t) \rightarrow \infty$ as t tends to infinity. This case is related to the card shuffling models of [28] and [11]; compare with [5].

2 The homogeneous Laplacian

In this section, we discuss general consequences of the homogeneity assumptions. First of all, the set of distances $\{d(x, y) : y \in X\}$ is the same for each $x \in X$. Since X is assumed to be non-compact, we have the following two cases.

Case 1. X is perfect, and $\{d(x, y) : y \in X\} = \{0\} \cup \{r_k : k \in \mathbb{Z}\}$, where $r_k < r_{k+1}$ with $\lim_{k \rightarrow \infty} r_k = \infty$ and $\lim_{k \rightarrow -\infty} r_k = 0$;

Case 2. X is countable, and $\{d(x, y) : y \in X\} = \{r_k : k \in \mathbb{N}_0\}$, where $r_0 = 0$, $r_k < r_{k+1}$ with $\lim_{k \rightarrow \infty} r_k = \infty$.

In both cases, we let

$$\mathcal{B}_k = \{B(x, r_k) : x \in X\}$$

be the collection of all closed balls of diameter r_k in X . This is a partition of X , and it is finer than \mathcal{B}_{k+1} . By homogeneity, all balls in \mathcal{B}_k are isometric. In particular, the number n_k of successor balls is the same for each ball in \mathcal{B}_k , where $k \in \mathbb{Z}$ in Case 1, and $k \in \mathbb{N}_0$ in Case 2. We call the two- or one-sided infinite sequence (n_k) the *degree sequence* of (X, d) . Note that $2 \leq n_k < \infty$.

When we pass from the *a priori* metric d to the intrinsic metric d_* given by (1.7), the distances r_k are transformed into new ones – which we shall denote by s_k – but the collections \mathcal{B}_k and the degree sequence remain the same.

We remark that one can associate an infinite *tree* with X . Its vertex set is \mathcal{B} , and there is an edge between any $B \in \mathcal{B}$ and its predecessor B' . In this situation, \mathcal{B}_k is the *horocycle* of the tree with index $-k$, and X is the (lower) *boundary* of that tree. For more details and figures in a context close to the one discussed here, see [6], as well as [16], [17], [14] and [7].

Independently of the initial algebraic or geometric model, our homogeneous ultrametric space (X, d) is uniquely determined by the degree sequence.

For having homogeneity, the reference measure m is also uniquely defined up to a constant factor. If we normalise by setting $m(B) = 1$ for each $B \in \mathcal{B}_0$, then *a fortiori*, for any $k \in \mathbb{Z}$ (Case 1), resp. $k \in \mathbb{N}_0$ (Case 2),

$$m(B) = V(k) \text{ for all } B \in \mathcal{B}_k, \text{ where } V(0) = 1, \\ V(k) = \begin{cases} n_1 n_2 \cdots n_k & \text{for } k > 0, \text{ and} \\ 1/(n_{k+1} n_{k+2} \cdots n_0) & \text{for } k < 0 \text{ in Case 1.} \end{cases}$$

This determines m uniquely as a measure on the Borel σ -algebra of X . Regarding the hierarchical Laplacian, homogeneity means that $C(B) = C_k$ is the same for each $B \in \mathcal{B}_k$.

Summarising, we see that any homogeneous ultrametric space plus Laplacian are completely determined by the degree sequence (n_k) and the sequence (C_k) which defines the homogeneous choice function.

Along with the choice function, also the eigenvalues of (1.2) depend only on k :

$$\lambda(B) = \lambda_k \text{ for all } B \in \mathcal{B}_k, \text{ where } \lambda_k = \sum_{\ell \geq k} C_\ell.$$

Each of them has infinite multiplicity in the pure point spectrum of L_C , see e.g [6, Section 3.2]. Regarding the intrinsic metric, recall that

$$d(x, y) = r_k \iff d_*(x, y) = s_k, \text{ where } s_k = 1/\lambda_k.$$

The volume function associated with the intrinsic metric is of course independent of $x \in X$, and given by

$$V_*(s) = m(B_*(x, s)) = V(k) \text{ for } s \in [s_k, s_{k+1}). \quad (2.2)$$

The spectral function $N(x, \tau) = N(\tau)$ of (1.8) is also independent of x , with $N(\tau) = 1/V_*(1/s)$. Thus, the formula (1.9) for the on-diagonal heat kernel becomes

$$\mathfrak{p}(t) = \int_0^\infty \exp(-t\tau) dN(\tau) = \sum_k e^{-t\lambda_k} \left(\frac{1}{V(k-1)} - \frac{1}{V(k)} \right). \quad (2.3)$$

Here and in the sequel, the summation ranges over $k \in \mathbb{Z}$ in Case 1, and over $k \in \mathbb{N}$ in Case 2.

The following plays an important role in the context of heat kernel estimates of many types of Laplacians, not just on ultrametric spaces. A monotone increasing function $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to satisfy the *doubling property* if there exists a constant $D > 0$ such that $F(2s) \leq D F(s)$ for all $s > 0$.

Proposition 2.1 *The following properties are equivalent.*

- (i) *The function $N(\tau)$ is doubling.*
- (ii) *There are finite bounds D and K such that for all $k \in \mathbb{Z}$, resp. $k \in \mathbb{N}$, and all $\tau > 0$,*

$$n_k \leq D \quad \text{and} \quad \#\{l : \tau \leq \lambda_l \leq 2\tau\} \leq K.$$

Proof. (i) \implies (ii). Assume that $N(\tau)$ is doubling. Since by definition $N(\tau) = 1/V_*(1/\tau)$, the function $V_*(s)$ is doubling as well. We use (2.2). Choose $s_k < s < s_{k+1}$ such that $2s > s_{k+1}$, then by the doubling property,

$$V_*(s_{k+1}) \leq V_*(2s) \leq D V_*(s) = D V_*(s_k).$$

It follows that

$$n_{k+1} = V_*(s_{k+1})/V_*(s_k) \leq D.$$

Let $\tau > 0$ and set $s = 1/(2\tau)$. Then there are k and r such that $s_k \leq s < s_{k+1}$ and $s_{k+r} \leq 2s < s_{k+r+1}$. We claim that

$$2^r \leq n_{k+1} \dots n_{k+r} \leq D^2.$$

Indeed, this is true if $r = 1$. Assuming that $r \geq 2$ we obtain

$$2^r \leq n_{k+1} \dots n_{k+r} \leq D \frac{V_*(s_{k+2})}{V_*(s_{k+1})} \dots \frac{V_*(s_{k+r})}{V_*(s_{k+r-1})} = D \frac{V_*(2s)}{V_*(s)} \leq D^2.$$

It follows that

$$\#\{l : \tau \leq \lambda_l \leq 2\tau\} = \#\{l : s \leq s_l \leq 2s\} \leq r + 1 \leq \frac{2 \ln D}{\ln 2} + 1.$$

(ii) \implies (i). Assume that $n_k \leq D$ for all $k \in \mathbb{Z}$, resp. $k \in \mathbb{N}$. For any $s > 0$, let $m(s) = \#\{l : s \leq s_l \leq 2s\}$. By assumption $m(s) \leq K$ for all $s > 0$. We have

$$V_*(2s) \leq D^{m(s)} V_*(s) \leq D^K V_*(s),$$

whence $V_*(\tau)$ is doubling. Since $N(\tau) = 1/V_*(1/\tau)$, the function $N(\tau)$ is doubling as well. \blacksquare

Remark 2.2 Recall that $\mathbf{p}(t)$ is the Laplace transform of the function $N(\tau)$. It follows that for all $t > 0$

$$c N(1/t) \leq \mathbf{p}(t) \leq c' N(1/t) \quad \text{for some } c, c' > 0 \quad (2.4)$$

if and only if the function $N(\tau)$ is doubling; see [6, Theorem 2.14 & Lemma 2.21]. (This holds also in the non-homogeneous case.)

When the function $N(\tau)$ is not doubling, one can state only that setting $M(\tau) = -\log N(\tau)$, we have for $t \rightarrow \infty$

$$\log \frac{1}{\mathfrak{p}(t)} \sim M^\star(t), \quad \text{where} \quad M^\star(t) = \inf\{t\tau + M(\tau) : \tau > 0\} \quad (2.5)$$

is the Legendre transform of the function $M(\tau)$. The papers [3] and [4] contain many computations based on (2.5). ■

Following [9, Section 2.2.2], a function $F(t) > 0$ is of *finite order* $\rho > 0$ if

$$\lim_{t \rightarrow \infty} \frac{\log F(t)}{\log t} = \rho. \quad (2.6)$$

Any function of the form $F(t) = a(t) t^\rho \exp(\log(1+t)^\varepsilon)$, where $0 < \varepsilon < 1$ and $a(t)$ is strictly positive and bounded, is of finite order ρ . Two examples related to the heat kernel $\mathfrak{p}(t)$ will be presented in Sections 3 and 4 of this paper.

Proposition 2.3 *The following statements are equivalent.*

- (a) *One of the functions $1/\mathfrak{p}(t)$, $1/N(1/t)$, $V_*(\tau)$ is of finite order ρ .*
- (b) *Each of the functions $1/\mathfrak{p}(t)$, $1/N(1/t)$, $V_*(\tau)$ is of finite order ρ .*
- (c) *$\log V(k) \sim \rho \log(1/\lambda_k)$ as $k \rightarrow \infty$.¹*

Proof. (a) \Rightarrow (b). The Abelian part of the statement

$$\text{order}(1/N(1/t)) = \rho \Rightarrow \text{order}(1/\mathfrak{p}(t)) = \rho$$

follows by a standard argument from the Laplace transform analysis. Thus, what is left is the Tauberian part of the statement, that is, the converse implication. Set again $M(\tau) = \log(1/N(\tau))$. By (2.6) and (2.5), for any given $\varepsilon > 0$ there exists $T > 0$ such that for all $t \geq T$,

$$t\tau + M(\tau) \geq \inf\{t\tau + M(\tau) : \tau > 0\} = M^\star(t) \geq (\rho - \varepsilon) \log t.$$

In particular, choosing $\tau = 1/t$ we obtain

$$M(1/t) \geq (\rho - \varepsilon) \log t - 1 \sim (\rho - \varepsilon) \log t.$$

¹ Throughout this paper, \sim denotes asymptotic equivalence, i.e., quotients tend to 1.

It follows that

$$\liminf_{t \rightarrow \infty} \frac{\log(1/N(1/t))}{\log t} \geq \rho - \varepsilon. \quad (2.7)$$

Let $\widetilde{M} \leq M$ be any continuous strictly decreasing function $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\widetilde{M}(0+) = \infty$. There exists $\widetilde{T} > 0$ such that for all $t \geq \widetilde{T}$,

$$\widetilde{M}^\star(t) \leq M^\star(t) \leq (\rho + \varepsilon) \log t. \quad (2.8)$$

For any given $t > 0$ there exists a unique τ_t such that $t \tau_t = \widetilde{M}(\tau_t)$, whence

$$\widetilde{M}^\star(t) \geq \min \left\{ \max \{ t \tau, \widetilde{M}(\tau) \} : \tau > 0 \right\} = t \tau_t.$$

This together with (2.8) implies that

$$t \tau_t = \widetilde{M}(\tau_t) \leq (\rho + \varepsilon) \log t.$$

In turn, this implies that $\tau_t \rightarrow 0$ as $t \rightarrow \infty$, and that for sufficiently large t ,

$$(\rho + \varepsilon) \log t \geq \widetilde{M} \left(\frac{\rho + \varepsilon}{t} \log t \right) \geq \widetilde{M} \left(\frac{1}{t^{1-\varepsilon}} \right),$$

whence, setting $\tau := t^{1-\varepsilon}$, we obtain

$$\widetilde{M} \left(\frac{1}{\tau} \right) \leq \frac{\rho + \varepsilon}{1 - \varepsilon} \log \tau.$$

As \widetilde{M} was chosen to be any continuous strictly decreasing function satisfying $\widetilde{M} \leq M$, it follows that

$$\limsup_{\tau \rightarrow \infty} \frac{\log(1/N(1/\tau))}{\log \tau} \leq \frac{\rho + \varepsilon}{1 - \varepsilon}.$$

This holds for arbitrarily small $\varepsilon > 0$, and together with (2.7), it leads to the desired result.

(b) \Leftrightarrow (c). By definition, $V_*(1/\lambda_k) = V(k)$, whence

$$\frac{V_*(1/\lambda_k)}{\log(1/\lambda_k)} = \frac{\log V(k)}{\log(1/\lambda_k)}.$$

Since $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$, the equivalence of (b) and (c) follows. \blacksquare

At last, recall that a positive function $F(t)$ varies regularly of index α if

$$\lim_{t \rightarrow \infty} \frac{F(\kappa t)}{F(t)} = \kappa^\alpha,$$

for all $\kappa \geq 1$. For example, each of the functions $t \mapsto t^\alpha$, $t^\alpha (\log t)^\beta$, $t^\alpha (\log t)^\beta (\log \log t)^\gamma$ varies regularly of index α , whereas $t \mapsto (2 + \sin t) t^\alpha$ does not vary regularly. See [9].

Proposition 2.4 *None of the functions $1/\mathfrak{p}(t)$, $1/N(1/t)$, $V_*(t)$ varies regularly.*

Proof. By Karamata's theory, the functions $1/\mathfrak{p}(t)$ and $1/N(1/t)$ vary regularly simultaneously. Since $V_*(t) = 1/N(1/t)$, this is true also for the functions $1/\mathfrak{p}(t)$ and $V_*(t)$.

Now assume by contradiction that the function $V_*(t)$ is regularly varying of index α , which evidently must be ≥ 0 , because $V_*(t)$ is increasing. With our notation $s_k = 1/\lambda_k$, choose $\varepsilon > 0$ and set $a = s_k - \varepsilon$ and $b = s_k + \varepsilon$. Since $V_*(s_k) = V(k)$, we have $V_*(a) \leq V(k-1)$ and $V_*(b) \geq V(k)$. As $s_k \rightarrow \infty$ and ε is fixed, $s_k + \varepsilon < (1 + \varepsilon)(s_k - \varepsilon)$ for large enough k , whence as $k \rightarrow \infty$,

$$2 \leq n_k = \frac{V(k)}{V(k-1)} \leq \frac{V_*(b)}{V_*(a)} \leq \frac{V_*((1 + \varepsilon)(s_k - \varepsilon))}{V_*(s_k - \varepsilon)} \rightarrow (1 + \varepsilon)^\alpha.$$

If ε is chosen small enough, this yields the contradiction we were looking for. ■

3 The operator of fractional derivative \mathfrak{D}^α

Let us for a moment return to the general setting of a not necessarily homogeneous ultrametric measure space (X, d, m) and an associated hierarchical Laplacian L_C of which we may assume without loss of generality that it is the standard Laplacian according to (1.5) and (1.6). Otherwise, we just replace d by the resulting intrinsic ultrametric (1.7).

Now take $\alpha > 0$ to introduce the new choice function

$$C_\alpha(B) = \left(\frac{1}{\text{diam}(B)} \right)^\alpha - \left(\frac{1}{\text{diam}(B')} \right)^\alpha, \quad B \in \mathcal{B}. \quad (3.1)$$

We denote the resulting operator by L_C^α . The corresponding eigenvalues are

$$\lambda^\alpha(B) = \left(\frac{1}{\text{diam}(B)} \right)^\alpha,$$

and the associated intrinsic ultrametric is d^α .

The space \mathcal{F} is the linear span of all $\mathbf{1}_B$, $B \in \mathcal{B}$. We can expand

$$\frac{\mathbf{1}_B}{m(B)} = \sum_{D \in \mathcal{B}: D \supseteq B} f_D,$$

a pointwise and $\mathcal{L}^2(X, m)$ -convergent series. Thus

$$L_C^\alpha \mathbf{1}_B = m(B) \sum_{D \in \mathcal{B}: D \supseteq B} \lambda^\alpha(D) f_D.$$

From this, we can infer the following.

Lemma 3.1 *For any $\alpha, \beta > 0$,*

$$L_C^\beta : \mathcal{F} \rightarrow \text{Dom}(L_C^\alpha) \quad \text{with} \quad L_C^\alpha \circ L_C^\beta = L_C^{\alpha+\beta} \quad \text{and} \quad (L_C^\alpha)^\beta = L_C^{\alpha\beta}.$$

Now let us briefly recall the construction of \mathbb{Q}_p for arbitrary $p \geq 2$: any non-zero element has the form

$$x = \sum_{n=-\infty}^{\infty} a_n p^n, \quad a_n \in \{0, \dots, p-1\}, \quad \exists k : a_k \neq 0 \text{ and } a_n = 0 \quad \forall n < k.$$

For such x , its p -adic (pseudo)norm is $\|x\|_p = p^{-k}$. The element 0 is represented by the series with all $a_n = 0$, and $\|x\|_p = 0$. In \mathbb{Q}_p , we have addition with carries, so that it extends the analogous operation for finite sums (i.e., when only finitely many a_n are $\neq 0$). We also have multiplication in the same sense, so that we get a ring with unit 1 (where $a_0 = 1$ and $a_n = 0$ for $n \neq 0$). The standard p -adic ultrametric is $\|x - y\|_p$.

We also have $\|x \cdot y\|_p \leq \|x\|_p \|y\|_p$, with equality when p is prime, and in this case, \mathbb{Q}_p is of course a field. The ring of p -adic integers is

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \|x\|_p \leq 1\}.$$

The set of non-zero distances is $\{r_k = p^k : k \in \mathbb{Z}\}$, and the closed ball of radius p^k around x is

$$B(x, p^k) = x + p^{-k} \mathbb{Z}_p.$$

Our reference measure m is Haar measure of the totally disconnected, locally compact abelian group $(\mathbb{Q}_p, +)$, normalised such that $m(\mathbb{Z}_p) = 1$. Thus,

$$V(k) = m(B(x, p^k)) = p^k.$$

The degree sequence (n_k) is constant, $n_k \equiv p$. Good references on general p -adic analysis (for prime p) are Katok [20] or Koblitz [21].

We now take the metric $d(x, y) = \|x - y\|_p / p$ and write $\mathfrak{D} = \mathfrak{D}_p$ for the standard hierarchical Laplacian associated with (\mathbb{Q}_p, d, m) . Then we consider the homogeneous operator $\mathfrak{D}^\alpha = \mathfrak{D}_p^\alpha$, $\alpha > 0$, according to the construction outlined at the beginning of this section. This is the operator of p -adic fractional derivative of order α in the terminology of [33], [34], [35]. For $f \in \mathcal{F}$ it can be written in the form

$$\mathfrak{D}^\alpha f(x) = \frac{p^\alpha - 1}{1 - p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{f(x) - f(y)}{\|x - y\|_p^{\alpha+1}} dm(y).$$

In terms of the Fourier transform,

$$\widehat{\mathfrak{D}^\alpha f}(\xi) = \|\xi\|_p^\alpha \widehat{f}(\xi), \quad \xi \in \mathbb{Q}_p,$$

which means that \mathfrak{D}^α is the spectral multiplier of order α ; compare with [6, §5.1] in the present setting. (This property also explains why the standard p -adic metric is divided by p here). In that form, it was first introduced and studied in [32] on \mathbb{Q}_p^d for arbitrary $d \geq 1$. For this reason, it is sometimes also called the *Taibleson Laplacian*.

The eigenvalues of \mathfrak{D}^α have the form

$$\lambda_k = \left(\frac{c}{V(k)} \right)^\alpha = \left(\frac{c}{p^k} \right)^\alpha, \quad (3.2)$$

with $c = p$. The value $c = 1$ is equally reasonable, and for our computations, the choice is not significant. The associated on-diagonal heat kernel with generic $c > 0$ is denoted $\mathfrak{p}_\alpha^{(c)}(t)$. In the special case of \mathfrak{D}^α , we just write $\mathfrak{p}_\alpha(t)$.

Theorem 3.2 *There exists a non-constant strictly positive continuous α -periodic function $\mathcal{A}(\tau) = \mathcal{A}_p(\tau)$ such that on \mathbb{Q}_p ,*

$$\mathfrak{p}_\alpha(t) = t^{-1/\alpha} \mathcal{A}(\log_p t).$$

For fixed $\alpha > 0$, as p tends to infinity,

$$\min \mathcal{A}_p \sim p^{-1}(\log p)^{1/\alpha} \quad \text{and} \quad \max \mathcal{A}_p \rightarrow (e\alpha)^{-1/\alpha}.$$

For general c in (3.2), the function $\mathcal{A}(\tau)$ has to be replaced by

$$(p/c) \mathcal{A}(\tau + \alpha \log_p(c/p)).$$

Proof. We give the proof for general $c > 0$ in (3.2). By (2.3),

$$\begin{aligned} \mathfrak{p}_\alpha^{(c)}(t) &= \sum_{k \in \mathbb{Z}} \exp\left(-t(cp^{-k})^\alpha\right) (p^{-(k-1)} - p^{-k}) \\ &= (p-1) \sum_{k \in \mathbb{Z}} \exp\left(-c^\alpha p^{\log_p t - \alpha k}\right) p^{(\log_p t - \alpha k)/\alpha} p^{-\log_p t/\alpha} \\ &= t^{-1/\alpha} \mathcal{A}^{(c)}(\log_p t), \end{aligned}$$

where

$$\mathcal{A}^{(c)}(\tau) = \frac{p-1}{c} \sum_{k \in \mathbb{Z}} \exp\left(-p^\tau - \alpha k + \alpha \log_p c\right) p^{(\tau - \alpha k + \alpha \log_p c)/\alpha}.$$

Setting $\mathcal{A}(\tau) = \mathcal{A}^{(p)}(\tau) = \frac{1}{p}\mathcal{A}^{(1)}(\tau)$ yields the proposed α -periodic continuous function and the transformation formula for generic c . It must be non-constant, since we know from Proposition 2.4 that $t \mapsto \mathbf{p}_\alpha(t)$ does not vary regularly at ∞ .

Deriving the asymptotics of $a_p = \min \mathcal{A}$ and $A_p = \max \mathcal{A}$ requires some laborious analysis. We simplify by performing the change of variables $r = p^\tau$, and define the two functions

$$f(r) = r^{1/\alpha} e^{-r} \quad \text{and} \quad g(r) = \sum_{k \in \mathbb{Z}} f(r p^{\alpha k}), \quad r \geq 0.$$

Then $\mathcal{A}(\tau) = \frac{p-1}{p} g(r)$, so that $a_p = \frac{p-1}{p} \min g$ and $A_p = \frac{p-1}{p} \max g$, both taken over the interval $[1, p^\alpha]$. The function $f(r)$ takes its strict maximal value $f(1/\alpha) = (e\alpha)^{-1/\alpha}$ at the unique stationary point $r = 1/\alpha$, and $f(r) \rightarrow 0$ as $r \rightarrow 0$, resp. $r \rightarrow \infty$.

We first claim that the dominant contribution to min and max comes from the central terms of the bi-infinite sum, that is,

$$g(r) = f(r) + f(r/p^\alpha) + O(1/p), \quad \text{as } p \rightarrow \infty \quad (3.3)$$

uniformly for $r \in [1, p^\alpha]$. Indeed, we write

$$g(r) = f(r) + f(r/p^\alpha) + \text{Sum}_I + \text{Sum}_{II},$$

where

$$\text{Sum}_I = \sum_{k \geq 1} f(r p^{\alpha k}) \quad \text{and} \quad \text{Sum}_{II} = \sum_{k \geq 2} f(r p^{-\alpha k}).$$

When p is sufficiently large then for all $k \geq 1$ and all $r \in [1, \infty)$

$$f(r p^{\alpha k}) \leq f(p^{\alpha k}) = p^k \exp(-p^{\alpha k}) \leq p^{-k},$$

because $f(r)$ is decreasing beyond $1/\alpha$. It follows that $\text{Sum}_I < 2/p$. Similarly, when p is sufficiently large, for all $k \geq 2$ and all $r \in [1, p^\alpha]$

$$f(r p^{-\alpha k}) \leq f(p^{-\alpha(k-1)}) < p^{-(k-1)}.$$

Therefore also $\text{Sum}_{II} < 2/p$. Thus, we are lead to study the function

$$h : [1, p^\alpha] \rightarrow \mathbb{R}_+, \quad h(r) = f(r) + f(r/p^\alpha)$$

dependig on p and α . By (3.3),

$$a_p = \min_{[1, p^\alpha]} h + O(1/p) \quad \text{and} \quad A_p = \max_{[1, p^\alpha]} h + O(1/p), \quad \text{as } p \rightarrow \infty.$$

We claim that as $p \rightarrow \infty$,

$$\min_{[1, p^\alpha]} h \sim \frac{(\log p)^{1/\alpha}}{p} \quad \text{and} \quad \max_{[1, p^\alpha]} h \rightarrow (e\alpha)^{-1/\alpha},$$

and this will complete the proof of the theorem. To prove this, we distinguish three cases.

Case 1. $0 < \alpha < 1$. Looking for the stationary points of $h(r)$, we transform the equation $h'(r) = 0$ into

$$\frac{r - \alpha^{-1}}{\alpha^{-1} - r p^{-\alpha}} = \frac{1}{p} \exp(r(1 - p^{-\alpha})). \quad (3.4)$$

Write $u(r)$ for the left hand side, and $v(r)$ for the right hand side. The denominator of $u(r)$ does not vanish in $[1, p^\alpha]$. Both functions are strictly increasing and strictly convex in that interval. Within the interval $[1, p^\alpha]$, for large p we find two solutions $r_p < s_p$ of (3.4). Namely, one sees that for any $\varepsilon > 0$, if p is large enough,

$$\begin{cases} 0 = u(\alpha^{-1}) < v(\alpha^{-1}), \\ u(\alpha^{-1} + \varepsilon) > v(\alpha^{-1} + \varepsilon), \\ u(\log p + \log \log p^{\alpha-\varepsilon}) > v(\log p + \log \log p^{\alpha-\varepsilon}), \\ u(\log p + \log \log p^{\alpha+\varepsilon}) < v(\log p + \log \log p^{\alpha+\varepsilon}). \end{cases}$$

Thus, we get $\alpha^{-1} < r_p < \alpha^{-1} + \varepsilon$ and $\log p + \log \log p^{\alpha-\varepsilon} < s_p < \log p + \log \log p^{\alpha+\varepsilon}$, whence

$$r_p \rightarrow \alpha^{-1} \quad \text{and} \quad s_p \sim s_p^* = \log p + \log \log p^\alpha, \quad \text{as } p \rightarrow \infty.$$

Let us show that there are no further solutions of (3.4) in $[1, p^\alpha]$: We compute $v' - u'$ and find that in our interval

$$\begin{aligned} & (u'(r))^{-1/2} - (v'(r))^{-1/2} \\ &= (1 - p^{-\alpha})^{1/2} \left((\alpha^{-1/2} - r \alpha^{1/2} p^{-\alpha}) - p^{1/2} \exp\left(-r \frac{1 - p^{-\alpha}}{2}\right) \right). \end{aligned} \quad (3.5)$$

For large p , the right hand side is < 0 at $r = 0$ and > 0 at $r = p^\alpha$. By strict convexity of the exponential function, there must be precisely one root of $v' - u'$ in $[1, p^\alpha]$, and it must be located between r_p and s_p . Now, if (3.4) had more than two solutions, then $v' - u'$ would have more than one root, which is not the case.

Tracing back the comparison between $u(r)$ and $v(r)$ to the sign of $h'(r)$, we see that the maximum of h in $[1, p^\alpha]$ is attained at r_p , and the minimum at s_p . Now a short asymptotic computation yields

$$\begin{aligned} \max_{[1, p^\alpha]} h &= h(r_p) \sim h(\alpha^{-1}) \sim f(\alpha^{-1}) = (e\alpha)^{-1/\alpha}, \quad \text{and} \\ \min_{[1, p^\alpha]} h &= h(s_p) \sim h(s_p^*) \sim f(s_p^*/p^\alpha) \sim \frac{(\log p)^{1/\alpha}}{p}, \end{aligned}$$

as proposed.

Case 2. $\alpha = 1$. In this case, we find an additional solution $t_p > s_p$ of (3.4) in $[1, p)$. Namely, for any $\varepsilon > 0$, if p is sufficiently large then

$$\begin{cases} u((1 - \varepsilon)p) < v((1 - \varepsilon)p), \\ \infty = u(p-) > v(p). \end{cases}$$

Now note that there can be no further roots of (3.4) in $[1, p]$, since otherwise $v' - u'$ would have more than two roots, which is impossible in view of (3.5).

We see that at r_p and t_p we have relative maxima of h , with $r_p \rightarrow 1$ and $t_p \sim p$ as $p \rightarrow \infty$. For large p , we have $h(r_p) > h(t_p)$, so that the absolute maximum is at r_p , and $h(r_p) \rightarrow e^{-1}$, as proposed. The minimum at s_p and the asymptotics for $h(s_p)$ are as in Case 1.

Case 3. $\alpha > 1$. In this case, the minimum at s_p and the corresponding asymptotics remain unchanged. To locate r_p (which is $> s_p$) it is better to invert both sides of (3.4) to get

$$\frac{\alpha^{-1} - r p^{-\alpha}}{r - \alpha^{-1}} = p \exp(-r(1 - p^{-\alpha})). \quad (3.6)$$

For any $\varepsilon \in (0, \alpha^{-1})$, if p is large enough then

$$\begin{cases} 0 = 1/u(\alpha^{-1}p^\alpha) < 1/v(\alpha^{-1}p^\alpha), \\ 1/u((\alpha^{-1} - \varepsilon)p^\alpha) > 1/((\alpha^{-1} - \varepsilon)p^\alpha). \end{cases}$$

Thus $(\alpha^{-1} - \varepsilon)p^\alpha < r_p < \alpha^{-1}p^\alpha$, and $r_p \sim \alpha^{-1}p^\alpha$. The argument to show that there are no solutions of (3.6) besides r_p and s_p is as in Case 1. We get once more

$$\max_{[1, p^\alpha]} h = h(r_p) \sim h(\alpha^{-1}p^\alpha) \sim f(\alpha^{-1}) = (e\alpha)^{-1/\alpha},$$

which concludes the proof. ■

We shall now generalise Theorem 3.2 in two directions: first, we replace equality in (3.2) by asymptotic equivalence. Second, we illustrate how the asymptotics of $\mathfrak{p}(t)$ depends only on the behaviour of the choice function (or equivalently, the eigenvalue function) for large balls, when $t \rightarrow \infty$, resp. for small balls, when $t \rightarrow 0$.

Theorem 3.3 *Let L_C be a homogeneous Laplacian with*

$$V(k) = p^k \quad \text{and} \quad \lambda_k \sim \begin{cases} (c_+/p^k)^\alpha, & \text{as } k \rightarrow +\infty, \\ (c_-/p^k)^\beta, & \text{as } k \rightarrow -\infty, \end{cases} \quad \alpha, \beta > 0.$$

Then there are constants $A_+ > a_+ > 0$ and $A_- > a_- > 0$ depending on p such that

$$\begin{aligned} \limsup_{t \rightarrow \infty} t^{1/\alpha} \mathfrak{p}(t) &= A_+ \quad \text{and} \quad \liminf_{t \rightarrow \infty} t^{1/\alpha} \mathfrak{p}(t) = a_+, \quad \text{while} \\ \limsup_{t \rightarrow 0} t^{1/\beta} \mathfrak{p}(t) &= A_- \quad \text{and} \quad \liminf_{t \rightarrow 0} t^{1/\beta} \mathfrak{p}(t) = a_-. \end{aligned}$$

Moreover, as $p \rightarrow \infty$, we have

$$\begin{aligned} A_+ &\sim p c_+^{-1} (e\alpha)^{-1/\alpha} \quad \text{and} \quad a_+ \sim c_+^{-1} (\log p)^{1/\alpha}, \quad \text{while} \\ A_- &\sim p c_-^{-1} (e\beta)^{-1/\beta} \quad \text{and} \quad a_- \sim c_-^{-1} (\log p)^{1/\beta}. \end{aligned}$$

Proof. We start with $t \rightarrow \infty$. Let $\lambda_k^+ = (c_+/p^k)^\alpha$. Take $\varepsilon > 0$ and choose $N = N_\varepsilon$ such that

$$(1 - \varepsilon)\lambda_k^+ \leq \lambda_k \leq (1 + \varepsilon)\lambda_k^+ \quad (3.7)$$

for all $k > N$. We decompose

$$\mathfrak{p}(t) = (p - 1) \left(\sum_{k \leq N} + \sum_{k > N} \right) e^{-t\lambda_k} p^{-k}.$$

We have $\lambda_k \geq \lambda_N$ for $k \leq N$, so that for $t \rightarrow \infty$,

$$\begin{aligned} t^{1/\alpha} (p - 1) \sum_{k \leq N} e^{-t\lambda_k} p^{-k} &\leq t^{1/\alpha} e^{-t\lambda_N/2} (p - 1) \sum_{k \leq N} e^{-t\lambda_k/2} p^{-k} \\ &\leq t^{1/\alpha} e^{-t\lambda_N/2} \mathfrak{p}(t/2) \leq t^{1/\alpha} e^{-t\lambda_N/2} \mathfrak{p}(1/2) \rightarrow 0. \end{aligned}$$

Writing $\mathfrak{p}_\alpha^{(c)}(t)$ for the heat kernel associated with the Laplacian of (3.2), we also have that

$$t^{1/\alpha} (p - 1) \sum_{k \leq N} e^{-t\lambda_k^+} p^{-k} \leq t^{1/\alpha} e^{-t\lambda_N^+/2} \mathfrak{p}_\alpha^{(c_+)}(1/2) \rightarrow 0, \quad \text{as } t \rightarrow \infty.$$

Now

$$\sum_{k>N} e^{-t(1+\varepsilon)\lambda_k^+} p^{-k} \leq \sum_{k>N} e^{-t\lambda_k} p^{-k} \leq \sum_{k>N} e^{-t(1-\varepsilon)\lambda_k^+} p^{-k}.$$

Thus, the preceding two estimates yield

$$\begin{aligned} (t(1+\varepsilon))^{1/\alpha} \mathfrak{p}_\alpha^{(c_+)}(t(1+\varepsilon)) + o(1) &\leq t^{1/\alpha} \mathfrak{p}(t) \\ &\leq (t(1-\varepsilon))^{1/\alpha} \mathfrak{p}_\alpha^{(c_+)}(t(1-\varepsilon)) + o(1), \quad \text{as } t \rightarrow \infty. \end{aligned}$$

This and Theorem 3.2 yield the proposed asymptotic behaviour at infinity.

To get the proposed asymptotic estimate when $t \rightarrow 0$, we now choose $M = M_\varepsilon$ such that (3.7) holds for all $k \leq M$. In this case,

$$t^{1/\beta} (p-1) \sum_{k>M} e^{-t\lambda_k} p^{-k} \leq t^{1/\beta} p^{-M} \rightarrow 0, \quad \text{as } t \rightarrow 0.$$

The same holds with λ_k^- in the place of λ_k . This time, we have

$$\sum_{k \leq M} e^{-t(1+\varepsilon)\lambda_k^-} p^{-k} \leq \sum_{k \leq M} e^{-t\lambda_k} p^{-k} \leq \sum_{k \leq M} e^{-t(1-\varepsilon)\lambda_k^-} p^{-k}.$$

We are lead to

$$\begin{aligned} (t(1+\varepsilon))^{1/\beta} \mathfrak{p}_\beta^{(c_-)}(t(1+\varepsilon)) + o(1) &\leq t^{1/\beta} \mathfrak{p}(t) \\ &\leq (t(1-\varepsilon))^{1/\beta} \mathfrak{p}_\beta^{(c_-)}(t(1-\varepsilon)) + o(1), \quad \text{as } t \rightarrow 0. \end{aligned}$$

This completes the proof. ■

Example 3.4 Theorem 3.3 applies, in particular, to the homogeneous hierarchical Laplacian $\mathfrak{D}^\alpha + \mathfrak{D}^\beta$ on \mathbb{Q}_p , where $0 < \alpha < \beta$. In this case, $c_+ = c_- = p$. ■

4 Random walks on ultrametric groups

We now recall the construction of our processes in terms of their semigroup $(P^t)_{t>0}$ of transition operators, as introduced and studied in [5] and [6]. On our non-compact ultrametric measure space (X, d, m) with the collection \mathcal{B} of balls with positive measure, we have the family of averaging operators

$$Q_B f(x) = \frac{1}{m(B)} \int_B f dm$$

which appear in the definition (1.3) of the hierarchical Laplacian. We take a probability measure σ supported by all of \mathbb{R}_+ and write $\sigma^t(r) = \sigma([0, r])^t$. With respect to the given metric d ,

$$P^t f(x) = \int_{\mathbb{R}_+} Q_{B(x,r)} f(x) d\sigma^t(r).$$

Then the infinitesimal generator of this Markov semigroup is the hierarchical Laplacian L_C , where the choice function $C(\cdot)$ is given via (1.2) by the eigenvalue function:

$$C(B) = \lambda(B) - \lambda(B') \quad \text{with} \quad \lambda(B) = \log(1/\sigma(\text{diam } B)), \quad B \in \mathcal{B}.$$

The transition operator $P = P^1$ gives rise to a discrete-time Markov chain (“isotropic random walk”) $(\mathcal{X}_n)_{n \geq 0}$ on X which is embedded in the continuous-time Markov process (\mathcal{X}_t) whose infinitesimal generator is L_C . The transition rule in one step is as follows: we first choose a random step length r according to the probability measure σ and then move to a point chosen according to m -uniform distribution within the (closed) ball of radius r around the current position in X .

Now let us consider the homogeneous situation. As noticed in [16], [17], the measure space (X, m) can then be identified with a locally compact totally disconnected group \mathfrak{G} equipped with its Haar measure. (Indeed, we may even identify it with an Abelian group.) Our homogeneous Laplacian L_C and the transition operator P are group-invariant, and (\mathcal{X}_n) is a random walk on that group. Write e for the group identity (0 in the Abelian case), and let (r_k) be the sequence of distances of (2.1). Then $\mathfrak{G}_k = B(e, r_k)$ is a compact-open subgroup of $\mathfrak{G} \equiv X$,

$$\mathfrak{G} = \bigcup_k \mathfrak{G}_k, \quad \text{and} \quad n_k = [\mathfrak{G}_k : \mathfrak{G}_{k-1}]$$

gives the degree sequence. The collection \mathcal{B}_k of balls with radius r_k consists of the left cosets of \mathfrak{G}_k in \mathfrak{G} . We usually normalise the Haar measure m such that $m(\mathfrak{G}_0) = 1$. In the discrete Case 2 of (2.1), $\mathfrak{G}_0 = \{e\}$ and m is the counting measure. The random walk (as well as the continuous-time process) is induced by the probability measure with density

$$\frac{d\mu}{dm} = \sum_k \frac{\pi_k}{m(\mathfrak{G}_k)} \mathbf{1}_{\mathfrak{G}_k}, \quad \text{where} \quad \pi_k = \sigma(r_{k+1}) - \sigma(r_k). \quad (4.1)$$

Here, k ranges over \mathbb{Z} in Case 1 and over \mathbb{N}_0 in Case 2 of (2.1). In this situation, for the step length distribution σ it is enough to specify the values π_k .

Example 4.1 In the case of \mathbb{Q}_p and the operator \mathfrak{D}^α , we have $\mathfrak{G}_k = p^{-k}\mathbb{Z}_p$, and with respect to the standard p -adic metric and Haar measure m with $m(\mathbb{Z}_p) = 1$, we have $r_k = p^k$ and

$$\pi_k = \exp(-p^{-\alpha k}) - \exp(-p^{-\alpha(k-1)}) \sim \begin{cases} (p^\alpha - 1)p^{-\alpha k}, & \text{as } k \rightarrow +\infty, \\ \exp(-p^{-\alpha k}), & \text{as } k \rightarrow -\infty. \end{cases}$$

The corresponding transition kernel asymptotics are then provided by Theorem 3.2, regardless of whether time is discrete or continuous.

The same also applies more generally to the Taibleson Laplacian with parameter α on \mathbb{Q}_p^d , which has the same degree and eigenvalue sequences as $\mathfrak{D}^{\alpha/d}$ on \mathbb{Q}_{p^d} , compare with [6, Subsection 5.3.2].

Indeed, the homogenous hierarchical Laplacians do not “see” the inner algebraic structure of each \mathfrak{G}_k . Thus, instead of \mathbb{Q}_p^d we might also take $X = (\mathcal{R}_p[t])^d$, where $\mathcal{R}_p[t]$ is the ring of formal Laurent series in the variable t over the ring $\mathcal{R}_p = \mathbb{Z}/(p\mathbb{Z})$ of integers modulo p . ■

Next, let us consider the situation where \mathfrak{G} is a discrete group, so that we are in Case 2 of (2.1). In this case, $\mathfrak{G}_0 = \{e\}$, and $(\mathfrak{G}_k)_{k \geq 0}$ is a strictly increasing sequence of finite subgroups of \mathfrak{G} . In other words, \mathfrak{G} is a countable, *locally finite* group (every finitely generated subgroup is finite). In this situation, $\mathbf{p}(t)$ is the probability that (\mathcal{X}_t) starting at the unit element is at e at time t . For integer time, $\mathbf{p}(n) = \mu^{(n)}(e)$ is the n -step return probability to the starting point, and $\mu^{(n)}$ is the n -th convolution power of μ .

The following is practically immediate from Theorem 3.3.

Theorem 4.2 *If $X = \mathfrak{G} = \bigcup_{k \geq 0} \mathfrak{G}_k$ as above, and the degree sequence $(n_k)_{k \geq 1}$ and the step length weights $(\pi_k)_{k \geq 0}$ satisfy*

$$n_k = [\mathfrak{G}_k : \mathfrak{G}_{k-1}] = p \geq 2 \quad \text{and} \quad \pi_k \sim \tilde{c} p^{-\alpha k} \quad \text{as } k \rightarrow \infty,$$

where $\tilde{c}, \alpha > 0$, then there are constants $A > a > 0$ such that

$$\limsup t^{1/\alpha} \mathbf{p}(t) = A \quad \text{and} \quad \liminf t^{1/\alpha} \mathbf{p}(t) = a,$$

as $t \rightarrow \infty$ in \mathbb{R}_+ or in \mathbb{N} . Moreover, as $p \rightarrow \infty$, we have

$$A \sim p (\tilde{c} e \alpha)^{-1/\alpha} \quad \text{and} \quad a \sim (\tilde{c} / \log p)^{-1/\alpha}.$$

Proof. We know that $\sigma_k = \sigma(r_k) \rightarrow 1$ as $k \rightarrow \infty$. Therefore

$$\lambda_k = -\log \sigma_k = -\log \left(1 - \sum_{j \geq k} \pi_j \right) \sim \sum_{j \geq k} \pi_j \sim \tilde{c} (1 - p^{-\alpha}) p^{-\alpha k} = (c/p^k)^\alpha,$$

where $c = (\tilde{c}(1 - p^{-\alpha}))^{1/\alpha}$. By (2.3),

$$\mathfrak{p}(t) = (p - 1) \sum_{k \geq 1} e^{-t\lambda_k} p^{-k}.$$

We are now in precisely the same situation as in the first part of the proof of Theorem 3.3, which leads to the desired result. \blacksquare

Remark 4.3 In this context the groups studied in [12] comprise

$$\mathfrak{G} = \bigoplus_{j=1}^{\infty} \mathbb{Z}(p)_j,$$

the direct sum of countably many copies $\mathbb{Z}(p)_j$ of the additive group $\mathbb{Z}(p) = \{0, 1, \dots, p - 1\}$ with addition modulo p . Thus, setting

$$\mathfrak{G}_k = \{(x_1, x_2, \dots) \in \mathbb{Z}(p)^{\mathbb{N}} : x_j = 0 \text{ for all } j > k\},$$

we have $\mathfrak{G} = \bigcup_k \mathfrak{G}_k$ with the resulting ultrametric structure described above. Each \mathfrak{G}_k can be identified with the direct product of k copies of $\mathbb{Z}(p)$. The random walks on \mathfrak{G} (and other infinite direct sums) considered in [12] are as follows: start with a symmetric probability measure μ_0 on $\mathbb{Z}(p)$, and let μ_k be the copy of μ_0 on the j -th coordinate in \mathfrak{G} , that is,

$$\mu_k(x_1, x_2, \dots) = \begin{cases} \mu_0(x_k), & \text{if } x_j = 0 \text{ for all } j \neq k, \\ 0, & \text{otherwise.} \end{cases}$$

Then let

$$\mu = \sum_k \pi_k \mu_k, \quad \text{where } \pi_k \geq 0, \sum_k \pi_k = 1.$$

This is not the same as our probability measure of (4.1), for which instead of μ_k we have uniform distribution on \mathfrak{G}_k . Indeed, the above μ is not related with a hierarchical Laplacian. Nevertheless, for values of π_k as in Theorem 4.2 (with equality instead of asymptotic equivalence, and stating only the case $p = 2$), Cartwright [12, Thm. 2] obtains asymptotics of the same type as ours and observes their periodic oscillation. However, he does not prove that the upper and lower bound in that oscillation are indeed distinct. \blacksquare

5 Isotropic random walks on the infinite symmetric group

In this final section, we study another class of cases that are part of the general setting of locally finite groups described in the preceding Section 4.

We consider the group $\mathfrak{S} = S_\infty$ of all permutations of \mathbb{N} which leave all but finitely many elements fixed. Here, $\mathfrak{S}_k = S_k$, the group of all permutations of $\{1, \dots, k\}$, here interpreted as those permutations of \mathbb{N} which fix each $n > k$. The balls in \mathcal{B}_k are the left cosets of S_{k+1} . Thus, the degree sequence and the resulting volume function are given by

$$n_k = k + 1 \quad \text{and} \quad V(k) = (k + 1)! \quad (5.1)$$

As in the previous sections, we consider a class of hierarchical Laplacians on S_∞ whose eigenvalues are a function of the volume (number of elements) of the balls. Consider the function

$$\Lambda : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad \Lambda(v) = v^{-\alpha} \phi(\log v), \quad (5.2)$$

where $\alpha > 0$ and the function $\phi(\cdot)$ varies regularly at ∞ . We assume that $\Lambda(\cdot)$ is strictly decreasing and that $\phi(\cdot)$ varies smoothly, which is no loss of generality for our asymptotic estimates [9, pages 44–55]. The eigenvalue sequence for the associated hierarchical Laplacian, as well as the weight sequence for the probability measure μ generating the associated random walk (in discrete or continuous time) according to (4.1) are then equivalently given by

$$\lambda_k = \Lambda(V(k)) \quad \text{and} \quad \pi_k = \tilde{\Lambda}(V(k)), \quad (5.3)$$

where $\tilde{\Lambda}$ is as Λ , with another smoothly varying function $\tilde{\phi}$ in the place of ϕ .

This should be compared with [28] and [11]. In [11], a similar random walk appears in the briefly mentioned “fifth model”, while the other random walks considered there do not arise from a hierarchical (\equiv isotropic) model. The probability weights used there are of the form $\pi_k \sim c/\Gamma(1 + k - \beta)$, where $\beta > 0$. They all correspond to the case $\alpha = 1$ of (5.2).

By Proposition 2.3 the function $1/\mathfrak{p}(t)$ is of finite order $1/\alpha > 0$ whereas by Proposition 2.1 $\mathfrak{p}(t)$ is not doubling, as opposed to the function $\mathfrak{p}_\alpha(t)$ of Section 3. Before stating our asymptotic result, which extends and strengthens the one in [8] substantially, we need to introduce the following positive real functions for sufficiently large t .

$$\begin{aligned} \rho(t) &= \frac{\log t}{\alpha \log \log t}, \\ \psi(t) &= t^{-1/\alpha} \left(\frac{\log \rho(t)}{\phi(\rho(t) \log \rho(t))} \right)^{1/\alpha} \quad \text{and} \\ \Psi(t) &= t^{-1/\alpha} \left(\frac{\rho(t)^\alpha}{e \alpha \phi(\rho(t) \log \rho(t))} \right)^{1/\alpha} \end{aligned} \quad (5.4)$$

Theorem 5.1 *For the hierarchical Laplacian, resp. isotropic random walk on S_∞ defined via (5.1), (5.2) and (5.3),*

$$\limsup \frac{\mathbf{p}(t)}{\Psi(t)} = 1 \quad \text{and} \quad \liminf \frac{\mathbf{p}(t)}{\psi(t)} = 1 ,$$

as $t \rightarrow \infty$ in \mathbb{R} or in \mathbb{N} , respectively.

Proof. Using the Euler gamma function, we consider

$$F(t, r) = t \Lambda(\Gamma(1+r)) + \log \Gamma(1+r) - \log r .$$

We start the labourious proof by using (2.3) to write

$$\mathbf{p}(t) = \sum_{k \geq 1} e^{-\lambda_k t} \left(\frac{1}{V(k-1)} - \frac{1}{V(k)} \right) \sim \sum_{k \geq M} e^{-\Lambda(k!)} \frac{k}{k!} = \sum_{k \geq M} e^{-F(t, k)} \quad (5.5)$$

for arbitrarily chosen (fixed) M , as $t \rightarrow \infty$. Indeed, the initial terms of the sum are exponentially smaller than the remainder of the series, as $t \rightarrow \infty$. We follow Laplace's method [9, Par. 4.12.8] to study the asymptotic behaviour of the rightmost series in (5.5).

Step 1. Localisation of the maximum. We look for the point $\bar{r}(t)$ where, for large t , the function $-F$ assumes its maximum in r . First of all, it follows from [1, Formula 6.3.16] that

$$\Phi(r) = \Gamma'(1+r)/\Gamma(1+r) \sim \log r \quad \text{as } r \rightarrow \infty ,$$

a strictly increasing function. Abbreviating $\bar{\Lambda}(v) = v\Lambda'(v)$, we compute the first partial derivative of F with respect to r :

$$F_r(t, r) = t \Phi(r) \bar{\Lambda}(\Gamma(1+r)) + \Phi(r) - 1/r .$$

Since $\Lambda(\cdot)$ varies smoothly along with ϕ , as $v \rightarrow \infty$

$$\bar{\Lambda}(v) = -\alpha \Lambda(v)(1 + o(1)) \quad \text{and} \quad v \bar{\Lambda}'(v) = \alpha^2 \Lambda(v)(1 + o(1)) . \quad (5.6)$$

Therefore, for arbitrary $t > 0$ we can choose M such that $r \mapsto F_r(t, r)$ is strictly increasing on $[M, \infty)$. Thus, $r \mapsto F(t, r)$ is strictly convex on that interval, and there is at most one stationary point which must solve the equation

$$-t \bar{\Lambda}(\Gamma(1+r)) = 1 - \frac{1}{r\Phi(r)} . \quad (5.7)$$

For any fixed t , when $r \rightarrow \infty$ the left hand side tends to 0, while the right hand side tends to 1. On the other hand, when t is large enough – say $t \geq t_M$

– then at $r = M$ the left hand side is strictly bigger than the right hand side. Therefore, there is a unique solution $\bar{r}(t)$ of (5.7), and this is where the minimum of $r \mapsto F(t, r)$ over $[M, \infty)$ is located. Smoothness of the functions appearing in (5.7) implies that $t \mapsto \bar{r}(t)$ is smooth. Furthermore, no matter how large M is chosen, the above shows that $\bar{r}(t) > M$ for $t \geq t_M$. Thus, $\bar{r}(t) \rightarrow \infty$ as $t \rightarrow \infty$. Also, $\bar{r}(\cdot)$ is strictly increasing. Set $\bar{v}(t) = \Gamma((1 + \bar{r}(t)))$. Then (5.6) yields that as $t \rightarrow \infty$,

$$t \Lambda(\bar{v}(t)) = \frac{1}{\alpha} + o(1), \quad \text{and} \quad \log \frac{1}{\Lambda(\bar{v}(t))} = \log(\alpha t) + o(1). \quad (5.8)$$

On the other hand, by (5.2) and Stirling's formula [1, Formula 6.1.41],

$$\log \frac{1}{\Lambda(\bar{v}(t))} \sim \alpha \log \bar{v}(t) \sim \alpha \bar{r}(t) \log \bar{r}(t), \quad \text{as } t \rightarrow \infty.$$

Now we see that $\alpha \bar{r}(t) \log \bar{r}(t) \sim \log(\alpha t) \sim \log t$, and

$$\log \bar{r}(t) \sim \log(\alpha \bar{r}(t) \log \bar{r}(t)) \sim \log \log t.$$

We conclude that

$$\bar{r}(t) \sim \rho(t) \quad \text{and} \quad e^{-F(t, \bar{r}(t))} \sim \Psi(t), \quad \text{as } t \rightarrow \infty. \quad (5.9)$$

The second asymptotic formula in (5.9) follows from the first one together with (5.8) and (5.2):

$$\begin{aligned} e^{-F(t, \bar{r}(t))} &\sim e^{-1/\alpha} \bar{r}(t) \bar{v}(t)^{-1} \sim e^{-1/\alpha} \bar{r}(t) \Lambda(\bar{v}(t))^{1/\alpha} \phi(\log \bar{v}(t))^{-1/\alpha} \\ &\sim \bar{r}(t) \left(e \alpha t \phi(\log \bar{r}(t) \log \bar{r}(t)) \right)^{-1/\alpha} \sim \Psi(t). \end{aligned}$$

The third of the last asymptotic equivalences relies on the Uniform Convergence Theorem for regularly varying functions [9, Thm. 1.5.2]. It will be used several times below without repeated mention.

Step 2. Asymptotic expansion near the maximum. We decompose $\bar{r}(t)$ into integer and fractional part: $\bar{r}(t) = \bar{k}(t) + \bar{\tau}(t)$. Our claim is that in (5.5), the main contribution to the sum comes from $\bar{k}(t)$ and $\bar{k}(t) + 1$, that is,

$$\mathfrak{p}(t) \sim \exp\left(-F(t, \bar{k}(t))\right) + \exp\left(-F(t, \bar{k}(t) + 1)\right) \quad \text{as } t \rightarrow \infty. \quad (5.10)$$

Indeed, we decompose the last the sum in (5.5):

$$\begin{aligned} \mathfrak{p}(t) &\sim e^{-F(t, \bar{k}(t))} (1 + A(t)) + e^{-F(t, \bar{k}(t)+1)} (1 + B(t)), \quad \text{where} \\ A(t) &= \sum_{k=M}^{\bar{k}(t)-1} e^{F(t, \bar{k}(t)) - F(t, k)} \quad \text{and} \quad B(t) = \sum_{k=\bar{k}(t)+2}^{\infty} e^{F(t, \bar{k}(t)+1) - F(t, k)} \end{aligned}$$

and show that $A(t), B(t) \rightarrow 0$ as $t \rightarrow \infty$.

We start with $B(t)$. Using that (λ_k) is strictly decreasing and $\bar{k}(t) + 1 > \bar{r}(t)$,

$$\begin{aligned} B(t) &= \sum_{k=\bar{k}(t)+1}^{\infty} e^{t(\lambda_{\bar{k}(t)} - \lambda_k)} \frac{\bar{k}(t)!}{k!} \\ &\leq \sum_{j=1}^{\infty} \frac{1}{(\bar{k}(t) + 1) \cdots (\bar{k}(t) + j)} \leq \sum_{j=1}^{\infty} \frac{1}{\bar{r}(t)^j} \rightarrow 0, \quad \text{as } t \rightarrow \infty. \end{aligned}$$

We turn to $A(t)$ and use that by (5.8), $t \Lambda(\bar{k}(t)!) \geq t \Lambda(\bar{v}(t)) \geq 1/(2\alpha)$ for all large t . We also use that $e^{-u} \leq \ell! u^{-\ell}$ for all $u > 0$ and all $\ell \in \mathbb{N}$. Thus,

$$\begin{aligned} A(t) &= \sum_{k=M}^{\bar{k}(t)-1} \exp\left(-t \Lambda(\bar{k}(t)!) \left(\frac{\Lambda(k!)}{\Lambda(\bar{k}(t)!)} - 1\right)\right) \frac{(\bar{k}(t) - 1)!}{(k - 1)!} \\ &\leq e^{1/(2\alpha)} \sum_{k=M}^{\bar{k}(t)-1} \exp\left(-\frac{1}{2\alpha} \frac{\Lambda(k!)}{\Lambda(\bar{k}(t)!)}\right) \frac{\bar{k}(t)!}{k!} \\ &\leq e^{1/(2\alpha)} \ell! (2\alpha)^\ell \sum_{k=M}^{\bar{k}(t)-1} \left(\frac{\Lambda(\bar{k}(t)!)}{\Lambda(k!)}\right)^\ell \frac{\bar{k}(t)!}{k!}. \end{aligned}$$

We now apply Potter's bounds to our regularly varying function $\Lambda(\cdot)$ with index $-\alpha$, see [9, Thm. 1.5.6]: possibly at the cost of taking a bigger value for M , given $\varepsilon = \alpha/2$ there is $a > 0$ such that for all k in our summation range,

$$\frac{\Lambda(\bar{k}(t)!)}{\Lambda(k!)} \leq a \left(\frac{\bar{k}(t)!}{k!}\right)^{-\alpha+\varepsilon}.$$

Therefore

$$A(t) \leq e^{1/(2\alpha)} \ell! (2c\alpha)^\ell \sum_{k=M}^{\bar{k}(t)-1} \left(\frac{\bar{k}(t)!}{k!}\right)^{1-\ell\alpha/2} \leq e^{1/(2\alpha)} \ell! (2a\alpha)^\ell \bar{k}(t)^{2-\ell\alpha/2}.$$

If we choose $\ell > 4/\alpha$, this tends to 0 as t and thus also $\bar{k}(t)$ tends to ∞ .

Step 3. The upper and lower asymptotic bounds. Let $\Delta_1(t) = t \Lambda(\bar{k}(t)!)$ and $\Delta_2(t) = t \Lambda((\bar{k}(t) + 1)!)$. We have from (5.2)

$$\begin{aligned} t^{1/\alpha} e^{-F(t, \bar{k}(t))} &= t^{1/\alpha} e^{-\Delta_1(t)} \bar{k}(t) \left(\frac{\Lambda(\Gamma(\bar{k}(t) + 1))}{\phi(\log \Gamma(\bar{k}(t) + 1))} \right)^{1/\alpha} \\ &\sim \Delta_1(t) e^{-\Delta_1(t)} \rho(t) \phi(\rho(t))^{-1/\alpha} \quad \text{and} \\ t^{1/\alpha} e^{-F(t, \bar{k}(t)+1)} &\sim \Delta_2(t) e^{-\Delta_2(t)} \rho(t) \phi(\rho(t))^{-1/\alpha}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathfrak{p}(t) &\sim \Psi(t) (e\alpha)^{1/\alpha} \left(f(\Delta_1(t)) + f(\Delta_2(t)) \right), \quad \text{where} \\ f(s) &= s^{1/\alpha} e^{-s}. \end{aligned} \quad (5.11)$$

Since $\Lambda(\cdot)$ is strictly decreasing, (5.8) and (5.2) yield that for large t ,

$$\Delta_2(t) < 1/\alpha \leq \Delta_1(t) \quad \text{and} \quad \delta(t) = \Delta_1(t)/\Delta_2(t) \sim \rho(t)^\alpha. \quad (5.12)$$

The function $f(s)$ takes its unique maximum $(e\alpha)^{-1/\alpha}$ at $s = 1/\alpha$ and tends to 0 at 0 and ∞ .

A. Upper bound. Suppose that t tends to infinity in such a way that $\Delta_1(t)$ remains bounded. Then $\Delta_2(t) \rightarrow 0$. Therefore

$$f(\Delta_1(t)) + f(\Delta_2(t)) \leq (e\alpha)^{-1/\alpha} + f(\Delta_2(t)) \rightarrow (e\alpha)^{-1/\alpha}.$$

On the other hand, suppose that t tends to infinity in such a way that $\Delta_1(t) \rightarrow \infty$. Then

$$f(\Delta_1(t)) + f(\Delta_2(t)) \leq f(\Delta_1(t)) + (e\alpha)^{-1/\alpha} \rightarrow (e\alpha)^{-1/\alpha}.$$

We infer that $\limsup_{t \rightarrow \infty} f(\Delta_1(t)) + f(\Delta_2(t)) \leq (e\alpha)^{-1/\alpha}$.

To see that this is an equality, we observe that $t \mapsto \bar{r}(t)$ is continuous and strictly increasing beyond some $t_0 > 0$. Therefore the (countable) set $T_0 = \{t > t_0 : \bar{r}(t) \in \mathbb{N}\} = \{t > t_0 : \bar{r}(t) = \bar{k}(t)\}$ is unbounded. If we let $t \in T_0$ tend to ∞ , then we know from Step 1 that $e^{-F(t, \bar{k}(t))} \sim \Psi(t)$, so that (5.10) and (5.11) imply

$$f(\Delta_1(t)) \sim (e\alpha)^{-1/\alpha} \quad \text{and} \quad f(\Delta_2(t)) \rightarrow 0.$$

The latter holds because by (5.8) we are in a regime where $\Delta_1(t) \sim 1/\alpha$ remains bounded, compare with the above. We now conclude that

$$\limsup_{t \rightarrow \infty} \mathfrak{p}(t)/\Psi(t) = 1.$$

B. Lower bound. We need to show that

$$\liminf_{t \rightarrow \infty} \frac{f(\Delta_1(t)) + f(\Delta_2(t))}{(\log \rho(t))^{1/\alpha} / \rho(t)} = 1. \quad (5.13)$$

Recall (5.12), and in particular the fact that $\Delta_1(t)$ and $\Delta_2(t)$ vary in $[1/\alpha, \infty)$ and $(0, 1/\alpha)$, respectively. Thus

$$\begin{aligned} f(\Delta_1(t)) + f(\Delta_2(t)) &\geq \max\{f(\Delta_1(t)), f(\Delta_2(t))\} \\ &\geq \min\left\{\max\{f(\delta(t)s), f(s)\} : s \in \left[\frac{1}{\alpha\delta(t)}, \frac{1}{\alpha}\right]\right\} \end{aligned}$$

On $[\frac{1}{\alpha\delta(t)}, \frac{1}{\alpha})$, the function $s \mapsto f(\delta(t)s)$ is strictly decreasing, while $f(s)$ is strictly decreasing. Therefore, the last minimum is attained in the unique point $s = s(t)$ which solves the equation $f(\delta(t)s) = f(s)$. We easily compute

$$s(t) = \frac{\delta(t) - 1}{\log \delta(t)^{1/\alpha}} \sim \frac{\rho(t)^\alpha}{\log \rho(t)} \quad \text{and} \quad f(s(t)) \sim \frac{(\log \rho(t))^{1/\alpha}}{\rho(t)},$$

as $t \rightarrow \infty$. This shows that the \liminf in (5.13) is at least 1. To verify that it is = 1, we first claim that the set

$$T_1 = \{t > t_0 : \Delta_1(t) = \log \bar{k}(t) + \log \log \bar{k}(t) + \log \alpha\}$$

is unbounded. To see this, we recall that by Stirling's formula, $\Gamma(r + \tau + 1) \sim r^\tau \Gamma(r + 1)$ as $r \rightarrow \infty$, uniformly for τ in any bounded interval. Using this, (5.2) and (5.8),

$$\frac{1}{\Gamma(\bar{k}(t) + 1)} \sim \frac{\bar{k}(t)^{\bar{\tau}(t)}}{\bar{v}(t)} = \frac{\Lambda(\bar{v}(t))^{1/\alpha} \bar{k}(t)^{\bar{\tau}(t)}}{\phi(\bar{v}(t))^{1/\alpha}} \sim \frac{\bar{k}(t)^{\bar{\tau}(t)}}{(\alpha t)^{1/\alpha} \phi(\bar{r}(t) \log \bar{r}(t))^{1/\alpha}},$$

so that

$$\Delta_1(t) \sim \frac{\phi(\Gamma(\bar{k}(t) + 1))}{\phi(\bar{r}(t) \log \bar{r}(t))} \alpha^{-1} \bar{k}(t)^{\alpha \bar{\tau}(t)} \sim \alpha^{-1} k(t)^{\alpha \bar{\tau}(t)}, \quad \text{as } t \rightarrow \infty.$$

Thus, the defining equation for the set T_1 becomes

$$\alpha^{-1} \bar{k}(t)^{\alpha \bar{\tau}(t)} (1 + o(1)) = \log \bar{k}(t) + \log \log \bar{k}(t) + \log \alpha,$$

which transforms into

$$\bar{\tau}(t) = \frac{\log \log \bar{k}(t)}{\alpha \log \bar{k}(t)} (1 + o(1)). \quad (5.14)$$

At this point, we recall from above the set $T_0 = \{t > t_0 : \bar{\tau}(t) = 0\}$, which is discrete, countable and unbounded, so that it has the form $\{t_j : j \in \mathbb{N}\}$ with $t_j < t_{j+1} \rightarrow \infty$. Since $\bar{r}(t)$ is continuous and strictly increasing, the function $\bar{\tau} : [t_j, t_{j+1}) \rightarrow [0, 1)$ is continuous, strictly increasing and surjective.

Therefore, if t_j is sufficiently large, the equation (5.14) has precisely one solution in $[t_j, t_{j+1})$, and T_1 is unbounded.

Now we can let $t \rightarrow \infty$ within T_1 , and then

$$\frac{f(\Delta_1(t))}{f(\Delta_2(t))} = \exp\left(\log(\delta(t)^{1/\alpha}) - \Delta_1(t)(1 - 1/\delta(t))\right) \rightarrow 0,$$

because we have $\delta(t) \sim \bar{k}(t)^\alpha$, so that

$$\log(\delta(t)^{1/\alpha}) - \Delta_1(t)(1 - 1/\delta(t)) \sim -\log \log \bar{k}(t).$$

We also see that $\Delta_2(t) \rightarrow 0$, so that

$$f(\Delta_1(t)) + f(\Delta_2(t)) \sim f(\Delta_2(t)) \sim \Delta_2(t)^{1/\alpha} \sim \frac{(\log \rho(t))^{1/\alpha}}{\rho(t)},$$

as $t \rightarrow \infty$ within T_1 . This proves (5.13). ■

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